CLASSICAL / QUANTUM THEORY OF 2-DIMENSIONAL HYDROGEN [‡]

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Why examine this toy system—or is it a toy? The classical 2-body problem, in its simplest form (no spin), asks for the time-dependence of twelve variables

$$x, y, z, p_z, p_y, p_z \text{ for each of two particles}$$

but by standard reduction leads promptly to an "equivalent one-body problem" in which only six variables

$$x, y, z, p_z, p_y, p_z$$

remain lively: those serve to describe the "motion, relative to the center of mass, of one hypothetical particle of reduced mass $m = \frac{m_1 m_2}{m_1+m_2}$." If

- externally impressed forces are absent, and if moreover
- the force of particle-particle interaction is central,

then the orbit is *confined necessarily to a plane*, and we find ourselves watching only four variables

$$x, y, p_x, p_z$$

Physics is rooted historically in one particular instance

$$H(p_x, p_y, x, y) = \tfrac{1}{2m}(p_x^2 + p_y^2) - \frac{k}{\sqrt{x^2+y^2}} \tag{1}$$

of the problem thus posed.

The reduced Kepler problem (1) relates to some real physics: it describes the paths along which the planets prefer to wander, fills the night sky with conic sections. But it presents only one subtle allusion to the fact that those plane figures are drawn on *cross-sections of three-dimensional space*: the implied force law

$$F \sim \tfrac{1}{r^2} \text{ becomes "geometrical"/intelligible only in Euclidean 3-space}$$

[‡] Text of a Reed College Physics Seminar presented on 3 February 1999.

Apart from that detail, the third dimension stands by as mere spectator while we pursue our celestial mechanical calculations. It becomes in this light a bit difficult to understand why the ancients became so preoccupied with spheres, since their experience presented them with only one fairly fanciful "celestial sphere," but a lot of inscribed planes—"circles," if you will.

I speak today of the "hydrogen problem" (Waßerstoff) rather than of the "Kepler problem" because I have in mind also its quantum mechanical variant. Atomic physics came into being as "celestial mechanics writ small" with the publication of Neils Bohr's "On the constitution of atoms and molecules."[1] A point to which I would draw attention is that Bohr—inventing what came to be called the "old quantum theory" as he went along, and taking (1) as his point of departure—*worked in the orbital plane*. For circular orbits classical mechanics supplies

$$\text{kinetic energy} = \tfrac{1}{2}\tfrac{L^2}{ma^2}$$

where a denotes the orbital radius and ma^2 is, in effect, a "moment of inertia." By the virial theorem[2] one has

$$\begin{aligned}\langle\text{kinetic energy}\rangle &= \tfrac{n+1}{2}\langle\text{potential energy}\rangle \quad \text{if } V(r) \sim r^{n+1} \\ &= -\tfrac{1}{2}\big\{-\tfrac{k}{a}\big\} \quad \text{in the hydrogenic case: } n = -2\end{aligned}$$

giving

$$a = \tfrac{L^2}{mk}$$

whence

$$\begin{aligned}E &= \langle\text{kinetic energy}\rangle + \langle\text{potential energy}\rangle \\ &= \tfrac{n+3}{2}\langle\text{potential energy}\rangle \\ &= -\tfrac{1}{2}\tfrac{mk^2}{L^2}\end{aligned}$$

and by formal substitution $L \mapsto n\hbar$ Bohr is led to the correct energy spectrum

$$E_n = -\tfrac{mk^2}{2\hbar^2}\tfrac{1}{n^2} \quad : \quad n = 1, 2, 3, \ldots \tag{2}$$

What makes this result so remarkable is that the "new quantum theory" would proceed from (1) to a Schrödinger equation

$$\Big\{-\tfrac{\hbar^2}{2m}\Big[\big(\tfrac{\partial}{\partial x}\big)^2 + \big(\tfrac{\partial}{\partial y}\big)^2\Big] - \tfrac{k}{\sqrt{x^2+y^2}}\Big\}\psi(x,y) = E\,\psi(x,y) \tag{3}$$

which, as we will see, yields an *incorrect spectral formula*:

$$E_n = -\tfrac{mk^2}{2\hbar^2}\tfrac{1}{(n-\frac{1}{2})^2} \quad : \quad n = 1, 2, 3, \ldots \tag{4}$$

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[1] Phil. Mag. **26**, 1, 476, 857 (1913).

[2] See H. Goldstein, *Classical Mechanics* (2nd edition, 1980), p. 85.

To reproduce and improve upon Bohr's success, Schrödinger himself *had to work in three dimensions*, from

$$\left\{-\tfrac{\hbar^2}{2m}\left[\left(\tfrac{\partial}{\partial x}\right)^2+\left(\tfrac{\partial}{\partial y}\right)^2+\left(\tfrac{\partial}{\partial x}\right)^2\right]-\tfrac{k}{\sqrt{x^2+y^2+z^2}}\right\}\psi(x,y,z)=E\psi(x,y,z) \qquad (5)$$

We come thus to the crooked conclusion that the Hamiltonian (1)

- refers classically to some real physics, but
- is quantum mechanically artificial.

The 3rd dimension is a classical spectator, but is quantum mechanically an active participant. We can understand this odd development on grounds that quantum mechanics is not a theory of orbits but a *field theory*, and the ψ-field samples the entire configuration space. That observation leaves unanswered, however, this seldom-asked question: *How did Bohr—armed only with his own toy anticipation of a quantum theory—manage to enjoy such success* (and what would have been the future of physics had events turned out otherwise)?

Today I will attempt to demonstrate that "2-dimensional hydrogen," toy system though it be, has nevertheless much of the first importance to teach us.

1. Classical background: Liouville systems & Euler's "problem of two centers". In 1849 Liouville drew attention to the fact that systems of the Lagrangian design

$$L=\tfrac{1}{2}u\cdot\left[\dot{q}_1^2+\dot{q}_2^2+\cdots+\dot{q}_n^2\right]-\frac{w_1(q_1)+w_2(q_2)+\cdots+w_n(q_n)}{u}$$
$$u\equiv u_1(q_1)+u_2(q_2)+\cdots+u_n(q_n)$$

give rise to equations of motion from which it is (by clever analysis) possible "by quadrature" (meaning "provided one can perform certain integrals") to obtain exact closed-form descriptions of the *trajectory* pursued by the system.[3] Those trajectories are classified by specification of

- the energy E;
- the values of certain separation constants $\left\{\epsilon_1,\epsilon_2,\ldots,\epsilon_n\right\}:\sum\epsilon_i=0$.

In the larger work from which this short account is taken I have shown that

- Liouville systems yield Hamilton-Jacobi equations which are invariably separable, but (trivialities aside)
- Liouville systems yield separable Schrödinger equations only in the 2-dimensional case.

Liouville's accomplishment made it possible to attribute Euler's successful analysis (1760–64) of the "2-dimensional two centers problem"

$$H(p_x,p_y,x,y)=\tfrac{1}{2m}(p_x^2+p_y^2)-\tfrac{k_1}{\sqrt{(x-a)^2+y^2}}-\tfrac{k_2}{\sqrt{(x+a)^2+y^2}}$$

[3] See E. T. Whittaker, *Analytical Dynamics* (1937), §43.

to the circumstance that the system acquires Liouville's form[4] when rendered in *confocal conic coordinates*, which are defined

$$\left.\begin{aligned} x &= a\cosh\xi\cos\eta \\ y &= a\sinh\xi\sin\eta \end{aligned}\right\} \tag{6}$$

and illustrated in Figure 1. It will be appreciated that the path pursued by a particle in the presence of *two* force centers is, in general, *not planar*;[5] Euler looked, therefore, to an unstable special case of the actual physical problem (and apologized for doing so... as also, a few years later, did Lagrange).

The Euler problem gives back the Kepler problem when one force center is turned off: $k_2 \downarrow 0$. This elementary observation carries the seldom-remarked implication that

> The 2-dimensional Kepler problem is separable in the sense of Liouville ("Liouville separable") in infinitely many confocal conic coordinate systems; namely, those with one focus at the force center (shifted origin) and the other anywhere (see Figure 2).

From which it follows more particularly that

> The 2-dimensional Kepler problem is Liouville separable in the "log-polar coordinate system"
>
> $$\left.\begin{aligned} x &= ae^s\cos\theta \\ y &= ae^s\sin\theta \end{aligned}\right\} \tag{7}$$
>
> into which all confocal conic coordinate systems degenerate when the foci coalesce: $a \downarrow 0$ (see Figure 3)

and, in the opposite limit, that

> The 2-dimensional Kepler problem is Liouville separable in the "parabolic coordinate systems"
>
> $$\left.\begin{aligned} x &= \tfrac{1}{2}(\mu^2 - \nu^2) \\ y &= \mu\nu \end{aligned}\right\} \tag{8}$$
>
> into which all confocal conic coordinate systems degenerate when one focus is installed at the origin and the other removed (in whatever direction) to infinity.

I have used Liouville's methods to establish (though the point is almost obvious) that *every Keplerean orbit can be identified with a curve-of-constant-coordinate in one or another of the confocal conic coordinate systems*.

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[4] For the demonstration, see §1 in my "Kepler problem by descent from the Euler problem" (Seminar Notes 1996).

[5] See Eli Snyder's thesis (1996) for display of some non-planar orbits.

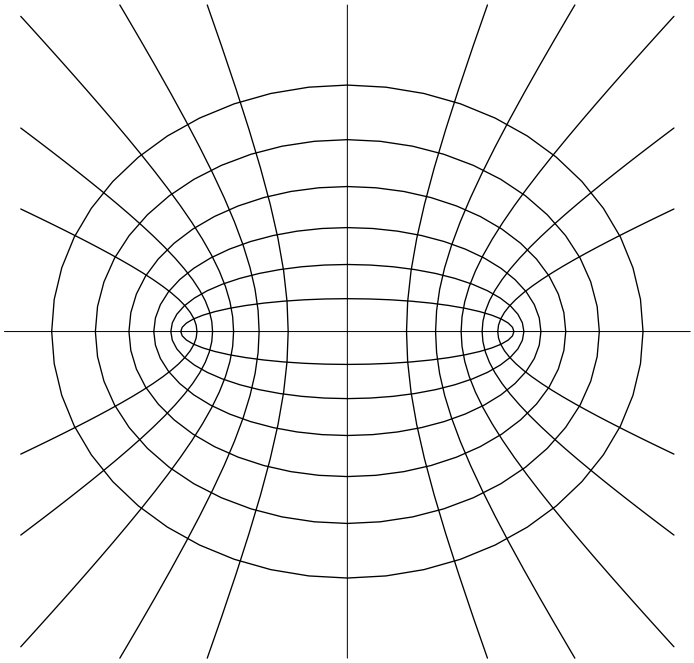

FIGURE 1: *Confocal conic coordinate system. ξ is constant on each ellipse; η is constant on each hyperbolic quadrant. The location of the foci is apparent. The figure was generated by (6) with the aid of Mathematica's* `ImplicitPlot` *package.*

Parabolic separation of the Kepler problem played, in fact, an important role in the history of quantum mechanics. For P. S. Epstein had noticed already in 1916 that if one Eulerean force center is made progressively stronger as it is removed to infinity, then one will be left at the stationary force center with a *parabolic description of a Keplerean system in the presence of a uniform field*. Epstein used this idea to construct, in language of the Old Quantum Theory, an account of the Stark effect, and his success made an impression upon the minds both of Schrödinger and of the young Wolfgang Pauli... with telling consequences, as will soon emerge.

2. Hidden symmetry in the classical theory. As we have seen, Liouville's method, in the n-dimensional case, yields $n-1$ separation constants and an associated population of "conserved observables:"

$$\epsilon_i = \text{conserved value of } G_i(p_1,\ldots,p_n,q_1,\ldots,q_n) \quad : \quad [H,G_i]=0$$
$$i = 1,2,\ldots,n-1$$

When the system yields to Liouville separation in several distinct coordinate systems one is led to correspondingly many distinct populations of conserved

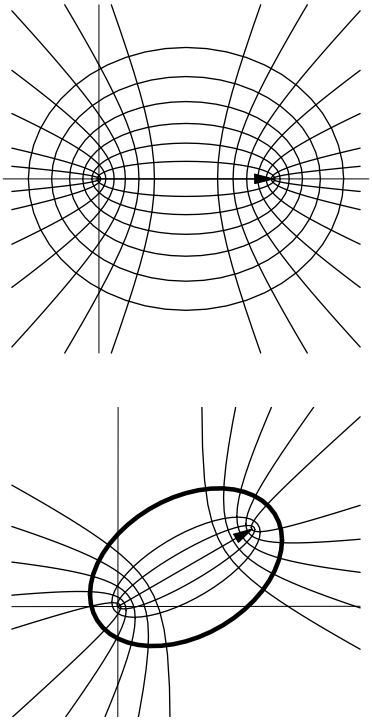

FIGURE 2: *Keplerean modification of the confocal conic coordinate system. The standard system has been first* **translated** *(top figure) so as to place a focus at the origin (force center), and then* **rotated** *(bottom figure). The heavy ellipse alludes fact that one can always identify Keplerean orbits with curves-of-constant-coordinate.*

observables. That, we expect, will be a rare occurance, but the 2-dimensional Kepler problem provides an example; one finds that

$$\begin{aligned}
\text{confocal conic separation} &\longrightarrow G = ma^2H + maK_x + \tfrac{1}{2}L_z^2 \\
&\quad\downarrow \\
\text{log-polar separation} &\longrightarrow G \sim\ L_z \equiv xp_y - yp_x \\
\text{parabolic separation} &\longrightarrow G \sim K_x \equiv \tfrac{1}{m}p_yL_z - kx\tfrac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Thus does Liouville's formulation of the Kepler problem lead automatically to

- the only component of $\boldsymbol{L} \equiv \boldsymbol{r} \times \boldsymbol{p}$ relevant to the 2-dimensional problem;
- one component of the Runge-Lenz vector $\boldsymbol{K} \equiv \frac{1}{m}(\,\boldsymbol{p} \times \boldsymbol{L}\,) - \frac{k}{r}\boldsymbol{r}$.

Angular momentum $\boldsymbol{L}$ was invented by Euler during the 1740's, and its conservation arises (Noether's theorem) as a expression of the overt rotational symmetry of a physical system; $L_z(p_x, p_y, x, y)$ is the Lie generator of canonical

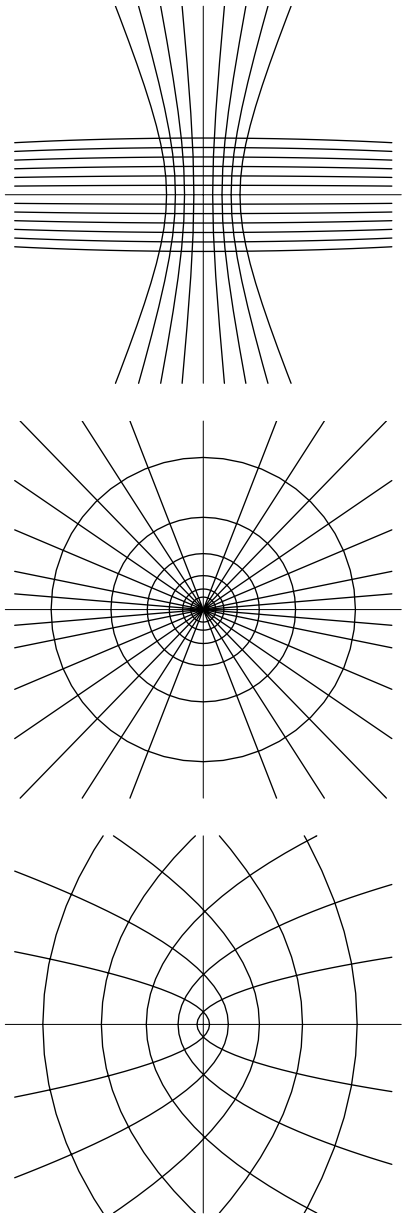

FIGURE 3: *Cartesian, log-polar and parabolic limits of the confocal conic coordinate system, got by*

- *magnifying the region near the origin (top),*
- *placing the foci very close together (middle)*
- *magnifying the region near the one focus (bottom).*

transformations within 4-dimensional phase space which, when projected onto the $\{x, y\}$-plane, acquire the form

$$\text{orbits} \longrightarrow \text{rotated orbits of the same energy}$$

The story of $\boldsymbol{K}$ is in several respects more interesting: the existence of such a Keplerean conservation law was known to Laplace already in 1799, and was rediscovered by Hamilton in 1845. Laplace's conserved observables (components of $\boldsymbol{K}$) were assembled into a vector for the first time by Gibbs, to demonstrate the utility of his new "vector analysis." Runge's contribution (1919) derived from Gibbs,' and was similarly expository, but led Lenz in 1924 to the first quantum mechanical application of Laplace's creation. Lenz's constribution is of enduring interest only because it engaged the attention of Pauli (see below). The observables $K_x(p_x, p_y, x, y)$ and $K_y(p_x, p_y, x, y)$ are generators of canonical transformations[6] which in projection acquire the form

$$\text{orbits} \longrightarrow \left\{ \begin{array}{l} \text{orbits of the same orientation and energy} \\ \text{but of altered eccentricity} \end{array} \right.$$

and are made possible this circumstance:

> The Kepler system and isotropic harmonic oscillator are the only systems endowed with the property that *all bounded orbits close upon themselves*.[7]

Given a population of conserved observables, it becomes natural to look for structure in the group of transformations which they serve collectively to generate. We therefore compute the Poisson brackets

$$\begin{aligned} [L_z, K_x] &= K_y \\ [K_y, L_z] &= K_x \\ [K_x, K_y] &= -\tfrac{2}{m} H \cdot L_z \end{aligned}$$

and notice that

- we have been led automatically to the "missing" component of $\boldsymbol{K}$, and
- by its inclusion we have achieved algebraic closure.

Within the "elliptic sector" of phase space (to which I henceforth confine my attention) $-\frac{2}{m}H$ can be manipulated as though it were a positive constant; we are therefore permitted to define

$$\left. \begin{aligned} J_x &\equiv K_x \big/ \sqrt{-\tfrac{2}{m}H} \\ J_y &\equiv K_y \big/ \sqrt{-\tfrac{2}{m}H} \\ J_z &\equiv L_z \end{aligned} \right\} \tag{9}$$

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[6] I do not know how to obtain $\boldsymbol{K}$ from Noether's theorem, and the detailed design of $\boldsymbol{K}$ leads me to think that such an effort would require fundamental *generalization* of Noether's theorem.

[7] This is the upshot of *Bertrand's theorem* (1873). Goldstein provides a detailed proof in his Appendix A (1980), but see also §§4.4/5 of J. L. McCauley, *Classical Mechanics: Transformations, Flows, Integrable & Chaotic Dynamics* (1997).

and to observe that

$$\left.\begin{aligned}[J_x, J_y] &= J_z \\ [J_y, J_z] &= J_x \\ [J_z, J_x] &= J_y\end{aligned}\right\} \tag{10}$$

mimic the commutation relations satisfied by the generators $\mathbb{J}_1$, $\mathbb{J}_2$ and $\mathbb{J}_3$ of the 3-dimensional rotation group $O(3)$. In the "hyperbolic sector" of phase space (where the scattered orbits live) one is led, by modification of the same argument, to the Lorentz group $O(1,2)$.

The $O(2)$ symmetry of the 2-dimensional Kepler system is overt, written in physical space for all to see. But $\{J_x, J_y, J_z\}$ generate within 4-dimensional phase space a representation of $O(3)$ which is covert—a "hidden symmetry."[8] We have encountered in interconnection amongst

- multiple separability,
- orbital closure, and
- hidden symmetry

which in its evident depth and obscure elegance is somewhat reminiscent of the interconnections which relate

- spin (whether integral or fractional),
- multiparticle wavefunction design (whether symmetric or antisymmetric),
- algebra of field operators (whether commutator or anticommutator).

It is a measure of the man that Pauli was a major player in the creation of *both* of those conceptual groupings.

3. 2-dimensional analog of Pauli's argument. In 1926—very shortly after the appearance of Heisenberg's "matrix mechanics," and prior to the appearance of Schrödinger's "wave mechanics"—Wolfgang Pauli (then 25 years old) published an algebraic theory of the hydrogen spectrum[9] of which Schiff gives a good account,[10] but which I describe now as it relates to "2-dimensional hydrogen."

We have

$$\mathsf{H} = \tfrac{1}{2m}\big[\,\mathsf{p}_x^2 + \mathsf{p}_y^2\,\big] - k\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}}$$

and the definitions

$$\begin{aligned}\mathsf{L}_z &= \mathsf{x}\,\mathsf{p}_y - \mathsf{y}\,\mathsf{p}_x \\ \mathsf{K}_x &\equiv +\tfrac{1}{2m}\big[\,\mathsf{p}_y\mathsf{L} + \mathsf{L}\,\mathsf{p}_y\big] - k\,\mathsf{x}\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}} \\ \mathsf{K}_y &\equiv -\tfrac{1}{2m}\big[\,\mathsf{p}_x\mathsf{L} + \mathsf{L}\,\mathsf{p}_x\big] - k\,\mathsf{y}\big[\,\mathsf{x}^2 + \mathsf{y}^2\,\big]^{-\frac{1}{2}}\end{aligned}$$

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[8] For real hydrogen, $O(3)$ is the overt symmetry, while the hidden symmetry is $O(4)$ on the elliptic sector, but the Lorentz group $O(1,3)$ on the hyperbolic sector.

[9] W. Pauli, "Uber das Waßerstoffspektrum vom Staandpunkt der neuen Quantenmechanik," Z. Physik **36**, 336 (1926).

[10] L. I. Schiff, *Quantum Mechanics* (3rd edition 1968), §30.

—the latter of which Pauli was content to borrow from Lenz. With the major assistance of *Mathematica* we establish the following commutation relations:

$$[\,\mathsf{H},\mathsf{L}_z\,]=[\,\mathsf{H},\mathsf{K}_x\,]=[\,\mathsf{H},\mathsf{K}_y\,]=\mathsf{O}$$

$$\begin{aligned}[\,\mathsf{L}_z\,,\mathsf{K}_x\,]&=+i\hbar\,\mathsf{K}_y\\ [\,\mathsf{K}_y,\,\mathsf{L}_z\,]&=+i\hbar\,\mathsf{K}_x\\ [\,\mathsf{K}_x,\mathsf{K}_y\,]&=-i\hbar\,\tfrac{2}{m}\mathsf{H}\,\mathsf{L}_z\end{aligned}$$

Within each eigenspace of H we can replace the operator by its eigenvalue E, which for bound states is negative. It makes sense, therefore, to mimic our classical practice, writing

$$\left.\begin{aligned}\mathsf{J}_1&\equiv\mathsf{K}_x\big/\sqrt{-\tfrac{2}{m}E}\\ \mathsf{J}_2&\equiv\mathsf{K}_y\big/\sqrt{-\tfrac{2}{m}E}\\ \mathsf{J}_3&\equiv\mathsf{L}_z\end{aligned}\right\}\tag{11}$$

in terms of which we have (compare (10))

$$\left.\begin{aligned}\big[\,\mathsf{J}_1,\mathsf{J}_2\,\big]&=i\hbar\,\mathsf{J}_3\\ \big[\,\mathsf{J}_2,\mathsf{J}_3\,\big]&=i\hbar\,\mathsf{J}_1\\ \big[\,\mathsf{J}_3,\mathsf{J}_1\,\big]&=i\hbar\,\mathsf{J}_2\end{aligned}\right\}\tag{12)}$$

These mimic the commutation relations satisfied by the familiar Pauli matrices (which were invented a couple of years later, and in another connection: the non-relativistic theory of spin), which are the generators not of $O(3)$ but of $SU(2)$, and support the "spinor representations" of the rotation group. Classical physics and quantum physics appear at this point to have diverged.

Returning to Pauli's line of argument: it follows immediately from (12) that $\big[\,\mathsf{J}^2,\mathsf{J}_1\,\big]=\big[\,\mathsf{J}^2,\mathsf{J}_2\,\big]=\big[\,\mathsf{J}^2,\mathsf{J}_3\,\big]=\mathsf{O}$ with

$$\begin{aligned}\mathsf{J}^2&\equiv\mathsf{J}_1^2+\mathsf{J}_2^2+\mathsf{J}_3^2\\ &=\mathsf{L}^2-\tfrac{m}{2E}\mathsf{K}^2\end{aligned}$$

But quantum mimicry[11] of the classical identity $K^2\equiv K_x^2+K_y^2=\frac{2}{m}HL^2+k^2$ gives

$$\mathsf{K}^2\equiv\mathsf{K}_x^2+\mathsf{K}_y^2=\tfrac{2}{m}\mathsf{H}\,(\,\mathsf{L}^2+\tfrac{1}{4}\hbar^2\mathsf{I}\,)+k^2\,\mathsf{I}$$

[11] This activity is much easier to talk about than to accomplish; the assistance of *Mathematica* is indispensable, but no "Symbolic Non-commutative Algebra" package is yet available. My improvisatory technique is clumsy, though vastly faster and more accurate than pen-and-ink.

so within each energy eigenspace we have

$$\begin{aligned}\mathsf{J}^2 &= \mathsf{L}^2 - \tfrac{m}{2E}\Big\{\tfrac{2}{m}E\,(\,\mathsf{L}^2 + \tfrac{1}{4}\hbar^2\mathsf{I}\,) + k^2\,\mathsf{I}\Big\} \\ &= -\big[\tfrac{1}{4}\hbar^2 + \tfrac{mk^2}{2E}\big]\mathsf{I} \end{aligned} \tag{13}$$

Borrowing now from the algebraic formulation of the 3-dimensional quantum theory of angular momentum[12] we know that

$$\mathsf{J}^2 \text{ has eigenvalues } j(j+1)\hbar^2 \text{ with } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

and that

$$\text{the eigenvalue } j(j+1)\hbar^2 \text{ is } (2j+1)\text{-fold degenerate}$$

Returning with this information to (13) we obtain

$$\begin{aligned} -\tfrac{mk^2}{2E} &= \big[j(j+1) + \tfrac{1}{4}\big]\hbar^2 \\ &= \tfrac{1}{4}(2j+1)^2\hbar^2 \end{aligned}$$

which can be written

$$\begin{aligned} E(j) &= -4\tfrac{mk^2}{2\hbar^2}\tfrac{1}{(2j+1)^2} \\ &= -4E_0\tfrac{1}{(\text{integer})^2} \end{aligned} \tag{14}$$

I reported at (4) that solution of the Schrödinger equation (3) yields

$$E = -4E_0\tfrac{1}{(\text{odd integer})^2} \quad \text{in the 2-dimensional case} \tag{15}$$

while Bohr/Pauli/Schrödinger obtained

$$E = -E_0\tfrac{1}{(\text{integer})^2} \qquad \text{in the 3-dimensional case}$$

Consistency of (14) with (13) requires that we must *exclude fractional j-values*: we must, in short, exclude precisely the representations which distinguish $SU(2)$ from $O(3)$. The classical/quantum physics of 2-dimensional hydrogen would be brought thus back again into agreement.[13]

But what principle serves to enforce such an exclusion? Exclusion of algebraically predicted states (of fractional angular momentum) is standardly accomplished by imposition of a requirement that the associated eigenfunctions be single-valued. I turn now, therefore, to discussion of the physical solutions of the Schrödinger equation.

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[12] See, for example, Schiff's §27, Mertzbacher's Chapter 16, or §§4.3 & 4.4 in D. Griffiths' *Introduction to Quantum Mechanics* (1995).

[13] It is now easy to understand how confusion on this point might arise, as historically it did arise. For detailed discussion and references see A. Cisneros & H. V. McIntosh, "Symmetry of the two-dimensional hydrogen atom," J. Math. Phys. **10**, 277 (1969).

4. Polar separation of the hydrogenic Schrödinger equation. In standard polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

the Schrödinger equation (3) reads

$$\Big\{-\tfrac{\hbar^2}{2m}\big[\tfrac{1}{r}\tfrac{\partial}{\partial r}r\tfrac{\partial}{\partial r}+\tfrac{1}{r^2}\big(\tfrac{\partial}{\partial\theta}\big)^2\big]-\tfrac{k}{r}\Big\}\psi = E\psi$$

which by $\psi = R(r)\cdot Y(\theta)$ separates to become

$$\Big\{\tfrac{1}{r}\tfrac{d}{dr}r\tfrac{d}{dr}+\tfrac{2m}{\hbar^2}\big[E+\tfrac{k}{r}\big]-\tfrac{\lambda}{r^2}\Big\}R(r) = 0$$
$$\big(\tfrac{d}{d\phi}\big)^2 Y(\theta) = -\lambda\cdot Y(\theta)$$

The argument proceeds now along familiar lines,[14] and leads to results which I summarize as they relate by "dimensional retraction"

$$\text{2-dimensions} \quad\longleftarrow\quad \text{3-dimensions}$$

to their 3-dimensional counterparts. First we have

$$\text{circular harmonics } Y_\ell(\theta) \quad\longleftarrow\quad \text{spherical harmonics } Y_\ell^m(\theta,\phi)$$

where $Y_\ell(\theta)\sim e^{i\ell\theta}$ are single-valued eigenfunctions of the angular momentum operator $\mathsf{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$: $\mathsf{L}_z Y_\ell(\theta) = \hbar\ell\cdot Y_\ell(\theta)$ with $\ell = 0,\pm1,\pm2,\ldots$ Insertion of $\lambda=\ell^2$ into the radial equation gives

$$R_{n\ell}(r)\sim e^{-\frac{1}{2}x}x^{|\ell|}L^{2|\ell|}_{n-|\ell|-1}(x) \quad\longleftarrow\quad R_{n\ell}(r)\sim e^{-\frac{1}{2}x}x^{\ell}L^{2\ell+1}_{n-\ell-1}(x)$$

which is function-theoretically modest on its face, but note: the dimensionless variable x acquires a retracted definition

$$x = \tfrac{2}{n-\frac{1}{2}}(r/a_0) \quad\longleftarrow\quad x = \tfrac{2}{n}(r/a_0) \quad:\quad a_0\equiv\tfrac{\hbar^2}{mk}=\text{"Bohr radius"}$$

which at $n=1$ reads $x = 4(r/a_0) \leftarrow x = 2(r/a_0)$; the radial function $R_{n\ell}(r)$ has been rendered *spatially more compact*. Which is a little surprising, for the Bohr model is, as I have pointed out, "effectively 2-dimensional," yet gives "orbits of the correct diameter" (by which we really mean only that it gives the observed spectroscopy). This development is consistent with

$$E_n = -E_0\tfrac{1}{(n-\frac{1}{2})^2} \quad\longleftarrow\quad E_n = -E_0\tfrac{1}{n^2} \quad:\quad n=1,2,3,\ldots$$

since tighter orbits should have lower energy.

[14] See B. Zaslow & M. E. Zandler, "Two-dimensional analog to the hydrogen atom," AJP **35**, 11118 (1967), but beware the many misprints.

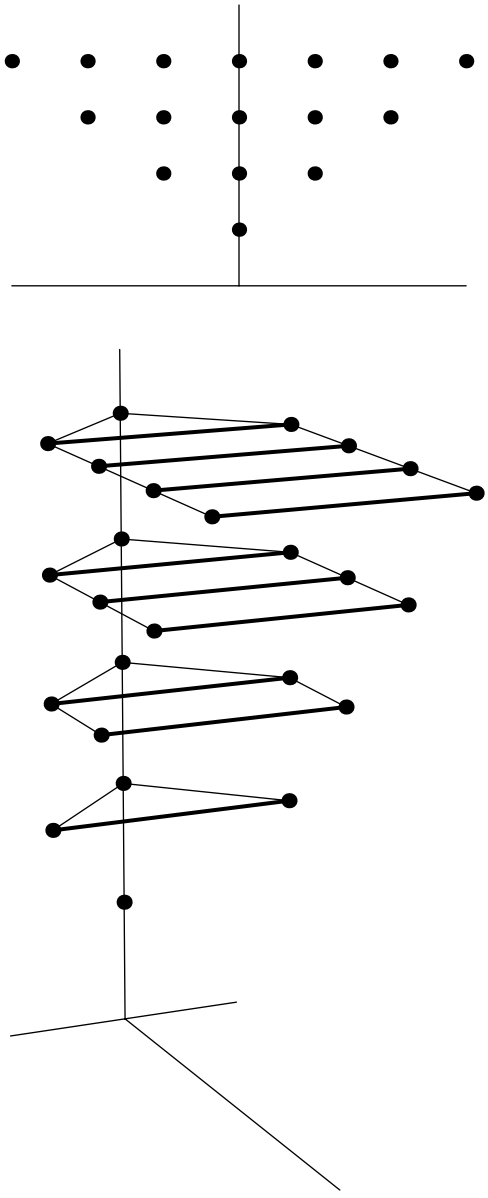

FIGURE 4: *Degeneracy of the energy spectrum of 2-dimensional hydrogen. In the upper display n runs $\uparrow$, ℓ runs $\leftrightarrow$. In the lower display—designed to mimic the virtues of Figure 5—n runs $\uparrow$, ℓ runs $\searrow$ and the $\swarrow\!\!\nearrow$ axis distinguishes states with respect to helicity (distinguishes $+\ell$ from $-\ell$).*

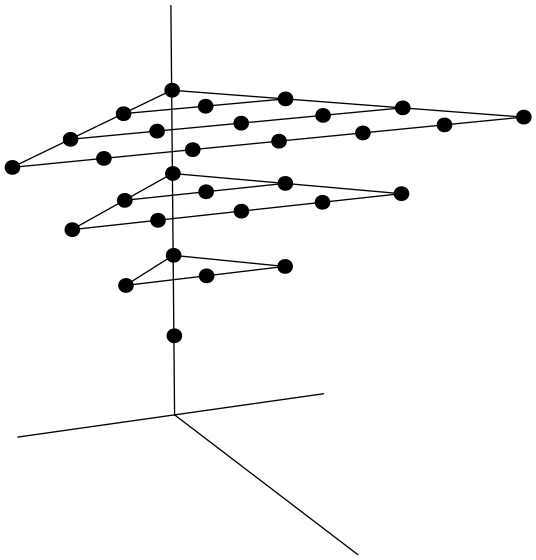

FIGURE 5: *Spectral degeneracy of 3-dimensional hydrogen. The principal quantum number n runs $\uparrow$, ℓ runs $\searrow$ and m runs $\swarrow\!\!\nearrow$. Symmetry with respect to rotations in physical 3-space accounts only for the equivalence of states of a given ℓ, which fold among themselves to yield $(2\ell+1)$-dimensional respresentations of $O(3)$. All n^2 states on the n^{th} tier fold among themselves to yield representations of the "accidental symmetry" group $O(4)$, with generators $\{\mathsf{L}_x, \mathsf{L}_y, \mathsf{L}_z, \mathsf{K}_x, \mathsf{K}_y, \mathsf{K}_z\}$. $O(4)$ is here an instance of a symmetry which arises not from overt geometrical considerations, but from the collective structure of the equations of motion (lives not in configuration space, but in phase space); such groups are called "dynamical groups."*

The preceding analysis yields simultaneous eigenfunctions of H and L_z, and accounts successfully for the enforced exclusion of the algebraically predicted states $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ But the argument exposes mysteries of its own: that the states $|n, +\ell)$ and $|n, -\ell)$ have the same energy follows transparently from the overt $O(2)$-symmetry of the physical system, but the analysis *leaves unexplained why states of different ℓ^2 should have the same energy*, as illustrated in Figure 4. "Accidental degeneracy" is present in the 2-dimensional hydrogen spectrum, just as it is present in the energy spectrum of real hydrogen. Such symmetry is a symptom of some "hidden symmetry" to which the analysis—which proceeds

without reference to **K**-conservation—does not allude.[15] Those are precisely the aspects of the problem which are illuminated when one looks to...

5. Parabolic separation of the hydrogenic Schrödinger equation. Schrödinger himself developed this topic already in 1926, and obtained 3-dimensional results which are reviewed in most of the better texts.[16] All authors emphasize the special suitability of parabolic separation when perturbations (Stark effect, Zeeman effect) install cylindrical symmetry in place of spherical symmetry, and also in connection with the desription of radiative/scattering processes. And all authors take their parabolic coordinates to be those defined in connection with Figure 6.

In 2-dimensional theory it proves advantageous, however, to respect the requirements of Liouville separability; we work, therefore, with the parabolic system introduced at (8), and are led straightforwardly from (3) to the separated equations

$$\begin{aligned}\big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\mu}\big)^2 - E\,\mu^2 - k_1 - \epsilon_1\big\}M(\mu) = 0\\ \big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\nu}\big)^2 - E\,\nu^2 - k_2 - \epsilon_2\big\}N(\nu) = 0\end{aligned}$$

in which $k_1 + k_2 = 2k$, $\epsilon_1 + \epsilon_2 = 0$. These equations can be written

$$\left.\begin{aligned}\big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\mu}\big)^2 + \tfrac{1}{2}m\omega^2\mu^2\big\}M(\mu) = (k_1+\epsilon_1)M(\mu)\\ \big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{d\nu}\big)^2 + \tfrac{1}{2}m\omega^2\nu^2\big\}N(\nu) = (k_2+\epsilon_2)N(\nu)\end{aligned}\right\} \qquad (16)$$

$$\tfrac{1}{2}m\omega^2 \equiv -E : \text{positive for bound states}$$

and—remarkably—place us in position to make formal use of the familiar *quantum theory of isotropic oscillators*; immediately

$$\begin{aligned}k_1 + \epsilon_1 &= \hbar\omega(n_1 + \tfrac{1}{2}) \quad : \quad n_1 = 0, 1, 2, \ldots\\ k_2 + \epsilon_2 &= \hbar\omega(n_2 + \tfrac{1}{2}) \quad : \quad n_2 = 0, 1, 2, \ldots\\ &\downarrow\\ 2k &= \hbar\omega(n_1 + n_2 + 1) \quad \text{giving} \quad E = -4\tfrac{mk^2}{2\hbar^2}\tfrac{1}{(n_1+n_2+1)^2}\end{aligned} \qquad (17)$$

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[15] One acquires interest at this point in adjustments—inclusion of relativistic effects, or moving farther down the first column of the periodic table—which might serve to *break the hidden symmetry without breaking the overt rotational symmetry* of the system.

[16] See, for example,
D. Bohm. *Quantum Theory* (1951),§58;
L. D. Landau & E. M. Lifshitz, *Quantum Mechanics* (1958), §37;
L. I. Schiff, *Quantum Mechanics* (3rd edition 1968), pp. 95–98;
E. Merzbacher, *Quantum Mechanics* (2nd edition 1970), pp. 245–250;
or H. A. Betha & E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (1957), §6.

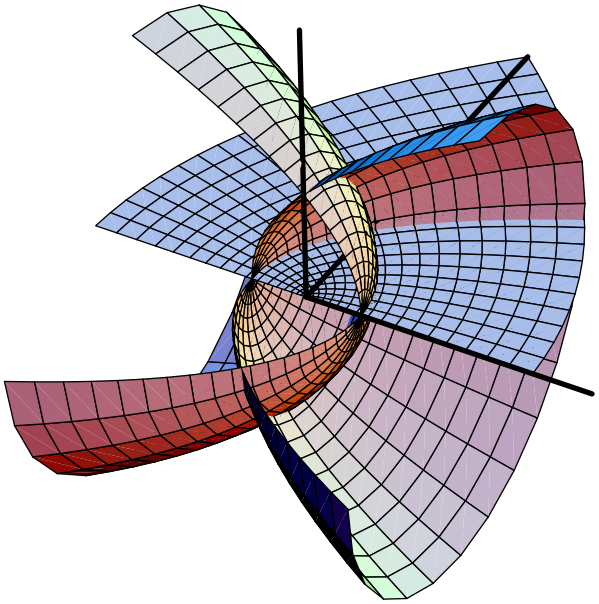

FIGURE 6: *Representation of the parabolic coordinate system*

$$x = \tfrac{1}{2}(\mu - \nu)$$
$$y = \sqrt{\mu\nu}\cos\phi$$
$$z = \sqrt{\mu\nu}\sin\phi$$

standard to 3-dimensional work. The figure shows a

- $\{\nu, \phi\}$*-coordinatized paraboloid of constant* μ *(opens left);*
- $\{\mu, \phi\}$*-coordinatized paraboloid of constant* ν *(opens right);*
- $\{\mu, \nu\}$*-coordinatized plane of constant* ϕ*.*

The Cartesian frame has been erected at the shared focus, which in the Keplerean application becomes the "force center."

which reproduces the upshot of (14), and presents the *same population of supernumerary spectral values as the spectrum to which Pauli's method led*.

Look, however, to the associated eigenfunctions: borrowing again from oscillator theory[17] we have

$$M_{n_1}(\mu) \sim e^{-\frac{1}{2}(\alpha\mu)^2} H_{n_1}(\alpha\mu)$$
$$N_{n_2}(\nu) \sim \text{similar}$$

where

$$\alpha \equiv \sqrt{\tfrac{m\omega}{\hbar}} = \big[-\tfrac{2mE}{\hbar^2}\big]^{\frac{1}{4}} = \tfrac{1}{\sqrt{a_0}}\big[\tfrac{2}{n_1+n_2+1}\big]^{\frac{1}{2}}$$

has (as required) the dimensionality of $(\text{length})^{-\frac{1}{2}}$ and—in stark contrast to the situation in oscillator theory—a meaning which is specific to each energy eigenspace. That important detail understood... we have been led by the "oscillator trick" to (unnormalized) hydrogenic energy eigenfunctions of the form

$$\Psi(\mu,\nu) = e^{-\frac{1}{2}\alpha^2(\mu^2+\nu^2)} H_{n_1}(\alpha\mu) H_{n_2}(\alpha\nu)$$

It is, however, an implication of (8) that

$$\big\{\mu,\nu\big\} \text{ and } \big\{-\mu,-\nu\big\} \text{ refer to the same point } \big\{x,y\big\}$$

We are obligated, therefore, to impose the single-valuedness condition

$$\Psi(\mu,\nu) = \Psi(-\mu,-\nu)$$

And that—by a familiar parity property of the Hermite polynomials—requires that n_1 and n_2 must be either both even or both odd, which in either case entails that

$$n_1 + n_2 + 1 \text{ must necessarily be odd}$$

Half of the wave functions supplied by the oscillator trick must therefore be discarded, and these are precisely the states responsible for the supernumerary spectral values, the states required to express the distinction between

- $SO(2)$, the known dynamical group of the 2-dimensional oscillator, and
- $O(3)$, the advertised dynamical group of the 2-dimensional hydrogen.[18]

Polar analysis led to eigenfunctions which can be developed as functions of μ and ν by means of

$$G_{j,\ell}(\mu,\nu) = \text{Exp}\Big[-\tfrac{\mu^2+\nu^2}{(2j+1)a_0}\Big] * \Big(\tfrac{2(\mu^2+\nu^2)}{(2j+1)a_0}\Big)^{\text{Abs}[\ell]}$$
$$* \,\text{LaguerreL}\Big[j - \text{Abs}[\ell], 2\,\text{Abs}[\ell], \tfrac{2(\mu^2+\nu^2)}{(2j+1)a_0}\Big]$$
$$* \,\text{ComplexExpand}\Big[\big(\tfrac{\mu^2-\nu^2}{\mu^2+\nu^2} + i\tfrac{2\mu\nu}{\mu^2+\nu^2}\big)^{\ell}\Big]$$

[17] See, for example, Schiff's §13.

[18] The point was first appreciated by Cisneros & McIntosh,[13] who also took parabolic separation as their point of departure.

while parabolic analysis gave

$$H_{n_1,n_2}(\mu,\nu) = \mathrm{Exp}\Big[-\tfrac{\mu^2+\nu^2}{(n_1+n_2+1)a_0}\Big] * \mathrm{HermiteH}\Big[n_1, \sqrt{\tfrac{2\mu^2}{(n_1+n_2+1)a_0}}\Big] * \mathrm{HermiteH}\Big[n_2, \sqrt{\tfrac{2\nu^2}{(n_1+n_2+1)a_0}}\Big]$$

Specifically, we have

$$G_{0,0}(\mu,\nu) = e^{-\frac{\mu^2+\nu^2}{a}}$$

$$\begin{aligned}
G_{1,+1}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} \big[2(\mu^2-\nu^2) + 4i\mu\nu\big] \\
G_{1,\ 0}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} \big[3a - 2(\mu^2+\nu^2)\big] \\
G_{1,-1}(\mu,\nu) &= \text{conjugate of } G_{1,+1}(\mu,\nu)
\end{aligned}$$

$$\begin{aligned}
G_{2,+2}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} \big[(4\mu^4 - 24\mu^2\nu^2 + 4\nu^4) + i(16\mu^3\nu - 16\mu\nu^3)\big] \\
G_{2,+1}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} \big[(4\mu^4 - 30a\mu^2 + 30a\nu^2 - 4\nu^4) \\
&\qquad + i(8\mu^3\nu - 60a\mu\nu + 8\mu\nu^3)\big] \\
G_{2,\ 0}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} \big[2(\mu^2+\nu^2)^2 - 20a(\mu^2+\nu^2) + 25a^2\big] \\
G_{2,-1}(\mu,\nu) &= \text{conjugate of } G_{2,+1}(\mu,\nu) \\
G_{2,-2}(\mu,\nu) &= \text{conjugate of } G_{2,+2}(\mu,\nu)
\end{aligned}$$

and

$$H_{0,0}(\mu,\nu) = e^{-\frac{\mu^2+\nu^2}{a}}$$

$$\begin{aligned}
H_{2,0}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} (8\mu^2 - 6a) \\
H_{1,1}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} (8\mu\nu) \\
H_{0,2}(\mu,\nu) &= \tfrac{1}{3a} e^{-\frac{\mu^2+\nu^2}{3a}} (8\nu^2 - 6a)
\end{aligned}$$

$$\begin{aligned}
H_{4,0}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} (64\mu^4 - 480a\mu^2 + 300a^2) \\
H_{3,1}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} (64\mu^3\nu - 240a\mu\nu) \\
H_{2,2}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} (64\mu^2\nu^2 - 80a\mu^2 - 80a\nu^2 + 100a^2) \\
H_{1,3}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} (64\mu\nu^3 - 240a\mu\nu) \\
H_{0,4}(\mu,\nu) &= \tfrac{1}{25a^2} e^{-\frac{\mu^2+\nu^2}{5a}} (64\nu^4 - 480a\nu^2 + 300a^2)
\end{aligned}$$

By inspection

$$\left.\begin{aligned}
G_{0,0} &= H_{0,0} \\
\\
G_{1,+1} &= \tfrac{1}{4}(H_{2,0} - H_{0,2}) + i\tfrac{1}{2}H_{1,1} \\
G_{1,\ 0} &= -\tfrac{1}{2}(H_{2,0} + H_{0,2}) \\
G_{1,-1} &= \tfrac{1}{4}(H_{2,0} - H_{0,2}) - i\tfrac{1}{2}H_{1,1} \\
\\
G_{2,+2} &= \tfrac{1}{16}(H_{4,0} + H_{0,4}) - \tfrac{3}{8}H_{2,2} + i\tfrac{1}{4}(H_{3,1} - H_{1,3}) \\
G_{2,+1} &= \tfrac{1}{16}(H_{4,0} - H_{0,4}) + i\tfrac{1}{8}(H_{3,1} + H_{1,3}) \\
G_{2,\ 0} &= \tfrac{1}{32}(H_{4,0} + H_{0,4}) + \tfrac{1}{16}H_{2,2} \\
G_{2,-1} &= \tfrac{1}{16}(H_{4,0} - H_{0,4}) - i\tfrac{1}{8}(H_{3,1} + H_{1,3}) \\
G_{2,-2} &= \tfrac{1}{16}(H_{4,0} + H_{0,4}) - \tfrac{3}{8}H_{2,2} - i\tfrac{1}{4}(H_{3,1} - H_{1,3})
\end{aligned}\right\} \qquad (18)$$

The short of it is this: polar analysis and parabolic analysis erect distinct bases within each of the respective energy eigenspaces; i.e., on each of the "branches" of the "state tree" shown in Figure 4 (lower). Those bases are interrelated by linear transformations (18) which, in the absence of normalization, we cannot expect to be unitary, but which are of a form

$$\text{(exponential)}\cdot\text{(power)}\cdot\text{(Laguerre)} = \sum \text{(gaussian)}\cdot\text{(Hermite)}\cdot\text{(Hermite)}$$

frequently encountered in the handbooks of higher analysis.

We confirm by calculation that the functions $G_{j,\ell}(\mu,\nu)$ are eigenfunctions of

$$\mathsf{L}_z = \tfrac{\hbar}{2i}\left[\mu\tfrac{\partial}{\partial\nu} - \nu\tfrac{\partial}{\partial\mu}\right] \quad : \quad \mathsf{L}_z G_{j,\ell} = \ell\hbar\cdot G_{j,\ell}$$

and that the functions $H_{n_1,n_2}(\mu,\nu)$ are eigenfunctions of

$$\mathsf{H} = -k\left[\tfrac{1}{2}a_0\nabla^2_{\text{parabolic}} + \tfrac{2}{\mu^2+\nu^2}\right]$$
$$\nabla^2_{\text{parabolic}} = \tfrac{1}{\mu^2+\nu^2}\left\{\left(\tfrac{\partial}{\partial\mu}\right)^2 + \left(\tfrac{\partial}{\partial\nu}\right)^2\right\}$$

Specifically, $\mathsf{H}\,H_{0,0} = -\mathcal{E}_0 H_{0,0}$ and

$$\mathsf{H}\begin{pmatrix} H_{2,0} \\ H_{1,1} \\ H_{0,2} \end{pmatrix} = -\ \tfrac{1}{9}\mathcal{E}_0 \begin{pmatrix} H_{2,0} \\ H_{1,1} \\ H_{0,2} \end{pmatrix}$$

$$\mathsf{H}\begin{pmatrix} H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = -\tfrac{1}{25}\mathcal{E}_0 \begin{pmatrix} H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix}$$

with $\mathcal{E}_0 = 2k/a_0 = 4{\cdot}mk^2/2\hbar^2$. The same can be said of the functions $G_{j,\ell}(\mu,\nu)$ which can, by (18), be assembled from these respective sets of eigenfunctions. All of which is reassuring, but hardly unexpected. More informative is the fact that computation based upon

$$\begin{aligned}
\mathsf{K}_x &= k\Big\{ -\tfrac{\mu^2-\nu^2}{2}\Big[\tfrac{1}{2}a_0\nabla^2 + \tfrac{2}{\mu^2+\nu^2}\Big] + \tfrac{1}{4}a_0\Big[\big(\tfrac{\partial}{\partial\mu}\big)^2 - \big(\tfrac{\partial}{\partial\nu}\big)^2\Big]\Big\} \\
\mathsf{K}_y &= k\Big\{ -\qquad \mu\nu\Big[\tfrac{1}{2}a_0\nabla^2 + \tfrac{2}{\mu^2+\nu^2}\Big] + \tfrac{1}{2}a_0\tfrac{\partial}{\partial\mu}\tfrac{\partial}{\partial\nu}\Big\}
\end{aligned}$$

gives

$$\mathsf{K}_x \begin{pmatrix} H_{0,0} \\ \\ H_{2,0} \\ H_{1,1} \\ H_{0,2} \\ \\ H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\frac{2}{3}H_{1,1} \\ -\frac{1}{3}(H_{2,0}+H_{0,2}) \\ -\frac{2}{3}H_{1,1} \\ \\ -\frac{4}{5}H_{3,1} \\ -\frac{1}{5}(3H_{2,2}+H_{4,0}) \\ -\frac{2}{5}(\ H_{3,1}+H_{1,3}) \\ -\frac{1}{5}(3H_{2,2}+H_{0,4}) \\ -\frac{4}{5}H_{1,3} \end{pmatrix} \qquad \mathsf{K}_y \begin{pmatrix} H_{0,0} \\ \\ H_{2,0} \\ H_{1,1} \\ H_{0,2} \\ \\ H_{4,0} \\ H_{3,1} \\ H_{2,2} \\ H_{1,3} \\ H_{0,4} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ -\frac{2}{3}H_{2,0} \\ 0 \\ +\frac{2}{3}H_{0,2} \\ \\ -\frac{4}{5}H_{4,0} \\ -\frac{2}{5}H_{3,1} \\ 0 \\ +\frac{2}{5}H_{1,3} \\ +\frac{4}{5}H_{0,4} \end{pmatrix}$$

We are brought thus to this pretty conclusion:

- polar separation yields simultaneous eigenfunctions $G_{j,\ell}$ of H and L_z;
- parabolic separation yields simultaneous eigenfunctions H_{n_1,n_2} of H and K_y.

The only surprise here is that we have encountered K_y where—for no good reason, when you think about it—we might have expected K_x.

6. Concluding remarks. "2-dimensional hydrogen" has revealed itself to be so valuable as a theoretical laboratory as to be, in my opinion, quite undeserving of its almost universal neglect. It mimics in every detail the formal depths of

- the celestial Kepler problem
- the quantum physics of real hydrogen,

but presents analytical problems which are at every turn simpler, and graphical opportunities which are invariably more immediate.

The energy spectrum of 2-dimensional hydrogen was found to be depressed relative to that of real hydrogen. Which posed a still-unresolved problem: *How did Bohr, with his essentially 2-dimensional hydrogen model, manage to obtain results in agreement with 3-dimensional observation?* Since, moreover, the ground state of a simple oscillator lies lower than that of an isotropic oscillator, and the ground state of a particle confined to an interval (length ℓ) lies lower than that of a particle confined to a box (area ℓ^2), we are led to ask: *Can it be shown that dimensional reduction invariably implies spectral depression?*

The Kepler problem is separable—actually separable in the sense of Liouville—in infinitely many coordinate systems (confocal conic coordinate

systems of every design); to speak of its "double separability" is rather grossly to underestimate the reality of the situation, but to emphasize a feature which the Kepler problem shares with the isotropic oscillator, a feature which appears to be tied up (but in a way which I do not know how to make precise) with universal orbital closure (Bertrand's theorem) and—more transparently—with the emergence of "hidden symmetry." In quantum mechanics the latter circumstance becomes manifest as "accidental degeneracy;" i.e., as a spectral degeneracy beyond that explicable by the overt geometrical considerations.[19] The theory of hydrogen serves at this point as a tutorial entry point to issues of latently far greater generality and importance.[20]

We were surprised to witness the emergence of the isotropic oscillator as a system not merely (in its orbital closure, double separability and hidden symmetry) analogous to but *directly relevant to* the hydrogenic system. The connection has been seen to be most vivid and direct when one works in parabolic coordinates, but is, of course, intrinsically coordinate-independent. In the much more elaborate work from which this report has been extracted[21] I show how ladder operators associated with the isotropic operator can be used to reconstruct the Keplerean Lenz operators which, as we have seen, permit one to "walk around" among the hydrogenic eigenstates.

I note finally that *ellipses are ellipses*, whether encountered in oscillators (which assign distinguished status to the center), planetary systems, atoms (which distinguish one focus at the expense of the other)... or polarized optical beams. The latter subject gave rise in 1852 to the "Stokes parameters" which were brought by Poincaré (1892) into contact with precisely the mathematics which has concerned us ($SU(2)$ and all that), and which in more recent times have given rise to an elegantly powerful technique for describing beam statistics and the beam-manipulation properties of optical devices. In some recent work[22] I have explored some of the mechanical applications of that body of theory.

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[19] Can one devise a classical analog of the quantum "spectrum" which is sharp enough to support a concept of "accidental degeneracy"?

[20] In the early spring of 1959 I directed the attention of Harold McIntosh, then a fellow graduate student at Brandeis University, to the recent appearance of S. P. Alliluev, "On the relation between "accidental" degeneracy and "hidden" symmetry of a system," Soviet Phys. JETP **6**, 156 (1958). Alliluev treats (*i*) n-dimensional hydrogen and (*ii*) the 2-dimensional oscillator in a way which suggested to McIntosh that "there is a lot Alliluev does not know." McIntosh thereupon sat down and very quickly wrote "On accidental degeneracy in classical and quantum mechanics," AJP **27**, 620 (1959), which has since become a minor classic in the field, and contains good references to the older literature. The term "dynamical group" was coined by A. O. Barut; see his *Dynamical Groups and Generalized Symmetries* (1971).

[21] "Reduced Kepler problem in elliptic coordinates" (1999).

[22] "Ellipsometry: Stokes' parameters & related constructions in optics & classical/quantum mechanics" (1996, revised 1999).